Math 251 Midterm 1

Name: ____________________

This exam has 11 questions, for a total of 100 points + 10 bonus points.

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	15	
2	10	
3	10	
4	10	
5	9	
6	10	
7	10	
8	6	
9	10	
10	10	
Total:	100	

Question	Bonus Points	Score
Bonus Question 1	10	
Total:	10	

Question 1. (15 pts)

(a) Find the domain of $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$

Solution: The domain of $f(x, y)$ is $\mathbb{R}^2$ except the origin $(0, 0)$, or equivalently,

$$\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$$

(b) Find the distance between the point $(1, 0, 1)$ and the plane

$$x + y + z = 1.$$

Solution: The distance

$$d = \frac{|1 + 0 + 1 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

(c) Give one example of hyperboloids with two sheets. (You only need to write its equation.)

Solution:

$$x^2 - y^2 - z^2 = 1$$

Question 2. (10 pts)

(a) Find an equation of the plane that passes through the points

$$A(1,0,1), \quad B(0,2,3), \quad C(-1,1,2)$$

Solution: Form vectors

$$\overrightarrow{AB} = \langle -1, 2, 2 \rangle, \quad \overrightarrow{AC} = \langle -2, 1, 1 \rangle$$

The plane has a normal vector

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, -3, 3 \rangle$$

Choose a point, say, $A(1,01)$, we have an equation of the plane

$$-3y + 3(z-1) = 0$$

or equivalently, $-3y + 3z = 3$.

(b) Find the line of intersection of the plane in part (a) with the plane $x + y - z = 1$.

Solution: To find the line of intersection, we need to find a point on the line and the direction of the line. The direction of the line is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 3 \\ 1 & 1 & -1 \end{vmatrix} = \langle 0, 3, 3 \rangle$$

To find a point on the line, we solve the following system:

$$\begin{cases} -3y + 3z = 3 \\ x - y - z = 1 \end{cases}$$

We get $x = 2, y = 0, z = 1$ is a solution. So $(2,0,1)$ is a point on the line of intersection. So the line of intersection has the following equations

$$x = 2, \quad y = 3t, \quad z = 1 + 3t$$

Question 3. (10 pts)

A curve is described by the vector function

$$\mathbf{r}(t) = \langle t^2 + \sin(2\pi t), \cos(\pi t), e^t \rangle.$$

(a) Find the derivative of $\mathbf{r}(t)$.

Solution:

$$\mathbf{r}'(t) = \left\langle 2t + 2\pi\cos(2\pi t), \quad -\pi\sin(\pi t), \quad e^t \right\rangle.$$

(b) Find the tangent line to this curve at the point $(1, -1, e)$.

Solution: First we need to figure out for which value of t the point $(1, -1, e)$ occurs, that is, to solve

$$\langle t^2 + \sin(2\pi t), \cos(\pi t), e^t \rangle = \langle 1, -1, e \rangle$$

We get $t = 1$.

Now plug $t = 1$ into the derivative from part (a), we get the tangent vector of the curve at the point $(1, -1, e)$.

$$\mathbf{v} = \langle 2 + 2\pi, 0, e \rangle$$

So the tangent line has equations:

$$x = 1 + (2 + 2\pi)t, \quad y = -1, \quad z = e + et$$

Question 4. (10 pts)

Show the following limit **does not** exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

Solution: Case 1: along the x-axis, that is, $y = 0$. We have

$$\lim_{x\to 0} 0 = 0$$

Case 2: along the direction $y = x^2$. We have

$$\lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Observe that $0 \neq \frac{1}{2}$. So we conclude that the limit does not exist.

Question 5. (9 pts)

Accept as a fact that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

Determine whether

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution: We have

(1)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

(2) $f(0,0) = 0 = \lim_{(x,y)\to(0,0)} f(x,y)$.

So the function f is continuous at $(0,0)$.

Question 6. (10 pts)

Let $z = x^2y + y^2$ with $x = u(\sin v)$ and $y = u^2 + e^v$. Find the value of $\dfrac{\partial z}{\partial v}$ for $(u, v) = (1, 0)$.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= (2xy)(u\cos v) + (x^2 + 2y)(e^v)\end{aligned}$$

For $(u, v) = (1, 0)$, we get $x = 1 \cdot \sin 0 = 0$ and $y = 1^2 + e^0 = 2$. Plug all these numbers into the expression of $\frac{\partial z}{\partial v}$, we have

$$\frac{\partial z}{\partial v}(1, 0) = 4$$

Question 7. (10 pts)

A surface is given by the graph of the following function

$$z = x^2 + xe^y + \sin y$$

Find the tangent plane of this surface when $(x, y) = (1, 0)$.

Solution: Set $f(x, y) = x^2 + xe^y + \sin y$. An equation of the tangent plane has formula

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Now compute

$$f_x = 2x + e^y$$

$$f_y = xe^y + \cos y$$

So we have

$$f_x(1, 0) = 3$$

$$f_y(1, 0) = 2$$

$$z_0 = f(1, 0) = 2$$

An equation of the tangent plane is given by

$$z - 2 = 3(x - 1) + 2y$$

Question 8. (6 pts)

Find all second partial derivatives of the function

$$f(x, y) = \sin(x^2 - y^2).$$

Solution: We need to find the first partial derivatives first.

$$\frac{\partial f}{\partial x} = 2x \cos(x^2 - y^2)$$

$$\frac{\partial f}{\partial y} = -2y \cos(x^2 - y^2)$$

Now the second partial derivatives are

$$f_{xx} = 2\cos(x^2 - y^2) - 4x^2 \sin(x^2 - y^2)$$

$$f_{xy} = f_{yx} = 4xy \sin(x^2 - y^2)$$

$$f_{yy} = -2\cos(x^2 - y^2) - 4y^2 \sin(x^2 - y^2)$$

Question 9. (10 pts)

Given the equation $y \cos z = x^3 e^{xyz} + x^2 y$, find $\partial z/\partial x$ by using implicit differentiation.

Solution: Set

$$F(x, y, z) = y \cos z - x^3 e^{xyz} - x^2 y$$

Then we have

$$\frac{\partial z}{\partial x} = \frac{-(\partial F/\partial x)}{(\partial F/\partial z)} = \frac{3x^2 e^{xyz} + x^3 yz e^{xyz} + 2xy}{-y \sin z - x^3(xy)e^{xyz}}$$

Question 10. (10 pts)

Use differentials to approximate the number $\sqrt{4.04} \cdot e^{0.01}$.

Solution: Set the function

$$f(x, y) = \sqrt{x} \cdot e^y$$

We shall compare $f(4.04, 0.01) = \sqrt{4.04} \cdot e^{0.01}$ with

$$f(4, 0) = \sqrt{4} \cdot e^0 = 2$$

Compute the differential of $f(x, y)$

$$df = f_x dx + f_y dy = (\frac{1}{2\sqrt{x}} e^y)dx + \sqrt{x} \cdot e^y dy$$

we have

$$f_x(4, 0) = \frac{1}{4}, f_y(4, 0) = 2$$

Moreover, $dx = 4.04 - 4 = 0.04$ and $dy = 0.01 - 0 = 0.01$. So we have

$$df = \frac{1}{4}(0.04) + 2 \cdot 0.01 = 0.03$$

Therefore,

$$\sqrt{4.04} \cdot e^{0.01} \approx f(4, 0) + df = 2 + 0.03 = 2.03$$

Bonus Question 1. (10 pts)

Determine whether the following limit exists or not.

$$\lim_{(x,y)\to(0,0)} \frac{y(\sin x)^2}{x^2+y^2}$$

Solution:

Here is one way to solve this problem: Observe that

$$\lim_{(x,y)\to(0,0)} \frac{y(\sin x)^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2+y^2}\frac{(\sin x)^2}{x^2}$$

We know

$$\lim_{x\to 0} \frac{(\sin x)^2}{x^2} = 1$$

So if we can solve the following limit

$$\lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2+y^2},$$

we can go back and solve the original question.

Now we in fact learned in class how to use the Squeeze theorem to show that

$$\lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2+y^2} = 0.$$

More precisely, we can use the inequality

$$0 \le \frac{|y|x^2}{x^2+y^2} \le |y|$$

and apply the Squeeze theorem.

To summarize, we see that

$$\lim_{(x,y)\to(0,0)} \frac{y(\sin x)^2}{x^2+y^2} = 0$$